

TWO-SIDED IDEALS IN THE RING OF DIFFERENTIAL OPERATORS ON A STANLEY-REISNER RING

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ABSTRACT. Let R be a Stanley-Reisner ring (that is, a reduced monomial ring) with coefficients in a domain k , and K its associated simplicial complex. Also let $D_k(R)$ be the ring of k -linear differential operators on R . We give two different descriptions of the two-sided ideal structure of $D_k(R)$ as being in bijection with certain well-known subcomplexes of K ; one based on explicit computation in the Weyl algebra, valid in any characteristic, and one valid in characteristic p based on the Frobenius splitting of R . A result of Traves [Tra99] on the $D_k(R)$ -module structure of R is also given a new proof and different interpretation using these techniques.

1. INTRODUCTION

Rings of k -linear differential operators $D_k(R)$ on a k -algebra R are generally difficult to study, even when the base ring R is well-behaved. Some descriptions of $D_k(R)$ are given in e.g. [Mus94] for the case of toric varieties, [Bav10a] and [Bav10b] for general smooth affine varieties (in zero and prime characteristic respectively), and [Tra99], [Tri97] and [Eri98] for Stanley-Reisner rings. Some criteria for simplicity of $D_k(R)$ exist (see [SVdB97] and [Sai07] among others), and the study of their left and right ideals, through the theory of D -modules, is well developed.

When $D_k(R)$ is not simple, however, it is an interesting problem to give a description of its *two-sided* ideals; the purpose of this paper is to do this for the case of Stanley-Reisner rings. Every Stanley-Reisner ring is the face ring R_K of a simplicial complex K , and we will give two different descriptions of the two-sided ideal structure of R in terms of the combinatorial structure of K ; namely the lattice of ideals is in a certain sense determined by the poset of subcomplexes of K that are *stars* of some face of K . The first description is based on explicit computations with monomials in the Weyl algebra, and the second (valid only in prime characteristic) takes advantage of the Frobenius splitting of R .

2. SOME PRELIMINARIES

Let us fix some notation. Throughout, k is a commutative domain. K will denote an abstract simplicial complex on vertices x_1, \dots, x_n ; we will not distinguish between K as an abstract simplicial complex and its topological realization. In the corresponding face rings (see 2.1) the indeterminate corresponding to a vertex x_i will also be named x_i to avoid notational clutter. Elements of K will be referred to as *simplices* or *faces*. For a face $\sigma \in K$, we let $x_\sigma := \prod_{x_i \in \sigma} x_i$. R will always mean

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a face ring R_K for a simplicial complex K . We use standard multiindex notation: x^a denotes $x_1^{a_1} \cdots x_n^{a_n}$, and $|a| = a_1 + \cdots + a_n$.

We briefly recall for the benefit of the reader some basics of Stanley-Reisner rings, omitting the proofs.

Definition 2.1. Let K be an abstract simplicial complex on vertices x_1, \dots, x_n . The *Stanley-Reisner ring*, or *face ring*, of K with coefficients in k is the ring $R_K = k[x_1, \dots, x_n]/I_K$, where $I_K = \langle x_{i_1} \cdots x_{i_r} \mid \{x_{i_1}, \dots, x_{i_r}\} \notin K \rangle$ is the ideal of square-free monomials corresponding to the non-faces of K , called the *face ideal* of K .

Geometrically, R_K is the coordinate ring of the cone on K , so $\dim R_K = \dim K + 1$. Accordingly, when we talk about support of elements, we will refer to faces of K when strictly speaking we mean the cones on these faces. If $K = \Delta_n$ is a simplex, I_K is the zero ideal, and R_K is the polynomial ring in n variables. If $K = K' * K''$ is the simplicial join of complexes K' and K'' , then $R_K \simeq R_{K'} \otimes_k R_{K''}$. Face rings are exactly the reduced monomial rings, i.e. quotients of polynomial rings by square-free monomial ideals.

Given a simplicial complex K , we will have use for a well-known class of subsets of K :

Definition 2.2. Let $\sigma \in K$ be a face. The *closed star* of σ in K is the subcomplex

$$st(\sigma, K) := \{\tau \in K \mid \tau \cup \sigma \in K\}.$$

The *open star* of σ in K is the set

$$st(\sigma, K)^\circ := \{\tau \in K \mid \sigma \cup \tau \in K \wedge \sigma \cap \tau \neq \emptyset\};$$

$st(\sigma, K)^\circ$ is the interior of $st(\sigma, K)$ in K , and $st(\sigma, K)$ is the closure of $st(\sigma, K)^\circ$ in K . The *open complement* of $st(\sigma, K)$ is the set (not usually a subcomplex)

$$U_\sigma(K) = K \setminus st(\sigma, K) = \{\tau \in K \mid \tau \cup \sigma \notin K\}.$$

Stars are important because the support of a principal monomial ideal of R_K , considered as an R_K -module, is exactly equal to the open star of some face, and the closed star is the smallest subcomplex containing it. For the remainder, we will take *star* to mean *closed star*. We will not have much need of comparing stars associated to different subcomplexes and so will often write simply $st(\sigma), U_\sigma$ if no confusion is likely to result. For completeness, we repeat a few simple facts:

- Lemma 2.3.**
- (i) If $\sigma \subset \tau$ are faces in K , $st(\sigma, K) \supset st(\tau, K)$;
 - (ii) If $L \subset K$ is a subcomplex containing σ , $st(\sigma, L) \subset st(\sigma, K)$;
 - (iii) For a face $\sigma = \tau \cup \{x\}$, $st(\sigma, K) = st(x, st(\tau, K))$.
 - (iv) $st(\tau) \subset st(\sigma)$ if and only if $\{\text{maximal simplices in } K \text{ that contain } \tau\} \subset \{\text{maximal simplices in } K \text{ that contain } \sigma\}$.
 - (v) $\sigma \in st(\tau) \Leftrightarrow \tau \in st(\sigma)$.
 - (vi) If $\sigma \cup \tau$ is a face of K , $st(\sigma)^\circ \cap st(\tau)^\circ = st(\sigma \cup \tau)^\circ$.

Proof. (i), (ii) and (v) are obvious. (iv) follows from the fact that a complex is determined by its maximal cells. (iii) follows from unwrapping the definitions:

$$(2.1) \quad st(x, st(\tau, K)) = \{\alpha \in st(\tau, K) \mid \alpha \cup \{x\} \in st(\tau, K)\}$$

$$(2.2) \quad = \{\alpha \in st(\tau, K) \mid \alpha \cup \{x\} \cup \tau \in K\}$$

$$(2.3) \quad = \{\alpha \in st(\tau, K) \mid \alpha \cup \sigma \in K\}$$

$$(2.4) \quad = st(\tau, K) \cap st(\sigma, K)$$

$$(2.5) \quad = st(\sigma, K)$$

where the last equality follows from (i). To show (vi), note that for any $\sigma \in K$, $st(\sigma)^\circ$ is the interior of the union of maximal simplices containing σ . It follows that $st(\sigma \cup \tau)^\circ$ is the interior of the union of maximal simplices containing both σ and τ , in other words the maximal simplices in $st(\sigma) \cap st(\tau)$. \square

We will need some properties of the face ideals $I_{st(\sigma)}$ and face rings $R_{st(\sigma)}$ of the subcomplexes $st(\sigma, K)$.

Lemma 2.4. (1) If K_1, K_2 are subcomplexes of K , $I_{K_1} + I_{K_2} = I_{K_1 \cup K_2}$ and $I_{K_1} \cap I_{K_2} = I_{K_1 \cap K_2}$.
 (2) $I_{st(\sigma)} = \langle x_\tau \mid \tau \in U_\sigma \rangle$.
 (3) The minimal primes of I_K are the face ideals $I_{st(\tau)}$ for the maximal simplices τ .

Proof. The first two items follow from the definition of $I_{st(\sigma)}$. For the last item, observe that $I_{st(\sigma)}$ is clearly prime when σ is a maximal simplex, as $I_{st(\sigma)} = \langle x_i \mid x_i \in U_\sigma \rangle$ and monomial ideals are prime exactly when they are generated by a subset of the variables; observe also that all $I_{st(\sigma)}$ are radical. These observations together with item 1 give the result, as $I_K = \bigcap_{\sigma \in K \text{ maximal}} I_{st(\sigma)}$. \square

We intend to study the ring of differential operators on R , so let us define what that is:

Definition 2.5. The ring $D_k(R)$ of k -linear differential operators on a k -algebra R is defined inductively by

$$D_k(R) = \bigcup_{n \geq 0} D_k^n(R)$$

where $D_k^0(R) = R$ and for $n > 0$, $D_k^n(R) := \{\phi \in \text{End}_k(R) \mid \forall r \in R : [\phi, r] \in D_k^{n-1}(R)\}$. Elements of $D_k^n(R) \setminus D_k^{n-1}(R)$ are said to have *order* n , and there is a natural filtration

$$D_k^0(R) \subset D_k^1(R) \subset D_k^2(R) \subset \dots$$

on $D_k(R)$ called the *order filtration*.

Definition 2.6. The *Weyl algebra* in n variables over k is the ring of differential operators on the polynomial ring $k[x_1, \dots, x_n]$. It is generated as an R -algebra by the *divided power operators* $\partial_i^{(a)} = \frac{1}{a!} \frac{\partial^a}{\partial x_i^a}$, with the relations $[x_i, x_j] = [\partial_i^{(a)}, \partial_j^{(b)}] = 0$ for $i \neq j$, $\partial_i^{(a)} \partial_i^{(b)} = \binom{a+b}{a} \partial_i^{(a+b)}$ and $[\partial_i^{(b)}, x_i] = \partial_i^{(b-1)}$ (in particular $[\partial_i, x_i] = 1$).

Remark 2.7. We use the divided power operators rather than the usual vector fields $\frac{\partial}{\partial x_i}$ as the latter do not generate the whole ring of differential operators in the case of characteristic p ; the divided power operators however always generate

everything regardless of the characteristic, as they define differential operators on \mathbb{Z} and so descend to any commutative ring. In characteristic zero, the derivations ∂_i suffice to generate everything; in characteristic p we need the full set of elements $\partial_i^{p^r}$ for $r \geq 0$, which suffice due to the relation $\partial_i^{(a)} \partial_i^{(b)} = \binom{a+b}{a} \partial_i^{(a+b)}$.

In the following, k will always be fixed, so we will omit it from the notation and write simply $D(R)$. Elements of k will be referred to as *constants*. One easily verifies that an element $x^a \partial^{(b)}$ in the Weyl algebra has order $|b|$.

3. THE TWO-SIDED IDEALS OF $D(R)$

When $R = R_K$ is a face ring, there exist several descriptions of $D(R)$ in the literature, see [Tri97], [Eri98] and [Tra99]. We wish to give a description of the two-sided ideals of $D(R)$ in terms of the combinatorics of K ; for our purposes, the following description due to Traves ([Tra99]) is the most convenient.

Theorem 3.1. *Let k be a commutative domain, and $R = k[X]/J$ a reduced monomial ring. An element $x^a \partial^{(b)} = \prod_i x_i^{a_i} \partial_i^{(b_i)}$ of the Weyl algebra over k is in $D(R)$ if and only if for each minimal prime \mathfrak{p} of R , we have either $x^a \in \mathfrak{p}$ or $x^b \notin \mathfrak{p}$. $D(R)$ is generated as a k -algebra by these elements, and they form a free basis of $D(R)$ as a left k -module.*

Example 3.2. Let $R = k[x_1, x_2, x_3]/(x_1 x_2 x_3)$. The associated simplicial complex K is the boundary of a 2-simplex. Then by 3.1, $D(R) = R \langle x_i^{a_i} \partial_i^{(b_i)} \mid a_i, b_i \in \mathbb{N} \rangle$.

Example 3.3. Let $R = k[x_1, x_2, x_3, x_4]/I$ where $I = (x_1 x_3, x_1 x_4, x_2 x_4)$. The associated complex K is a chain of three 1-simplices, connected in order x_1, x_2, x_3, x_4 . Theorem 3.1 gives $D(R) = R \langle x_1^a \partial_1^{(b)}, x_2^a \partial_2^{(b)}, x_3^a \partial_3^{(b)}, x_4^a \partial_4^{(b)}, x_1^a \partial_2^{(b)}, x_4^a \partial_3^{(b)} \rangle$ (for $a, b > 0$).

Note that in both examples, generators of the form $x_i^a \partial_i^{(b)}$ appear; it is not hard to see that such “toric” operators are always in $D(R)$. In 3.3, we also have generators of the form e.g. $x_i^a \partial_j^{(b)}$ (where $i \neq j$). To understand when this happens, we may give a somewhat more geometric formulation of 3.1:

Proposition 3.4. *Let K be a simplicial complex and $R = R_K$ its face ring. Also let $x^a = \prod x_i^{a_i}, x^b = \prod x_j^{b_j}$ be such that $\text{supp}(x^a) = \text{st}(\sigma)$ and $\text{supp}(x^b) = \text{st}(\tau)$, for some $\sigma, \tau \in K$. Then $x^a \partial^{(b)} = \prod_i x_i^{a_i} \partial_i^{(b_i)}$ is in $D(R)$ if and only if $\text{st}(\sigma) \subset \text{st}(\tau)$.*

Proof. Let P_{x^a} denote the set of minimal primes in R that contain x^a , and $P_{\neg x^a}$ the set of minimal primes that does not contain x^a . Clearly, $P_{x^a} \cup P_{\neg x^a}$ is equal to the set of minimal primes in R ; denote this by P . Recalling from 2.4 that the minimal primes of R are the face ideals $I_{\text{st}(\alpha)}$ for maximal simplices α , we can reformulate these definitions: P_{x^a} is the set of ideals $I_{\text{st}(\alpha)}$ such that α is maximal and $x^a \in I_{\text{st}(\alpha)}$, in other words those ideals $I_{\text{st}(\alpha)}$ such that α is maximal and $\alpha \in U_\sigma$; and $P_{\neg x^a}$ is the set of ideals $I_{\text{st}(\alpha)}$ with α maximal and contained in $\text{st}(\sigma)$. Again using 2.4, the ideal $I_{\text{st}(\sigma)}$ defining $\text{st}(\sigma)$ is equal to the intersection of all ideals in $P_{\neg x^a}$. Unwrapping definitions, we get

$$\begin{aligned} \text{st}(\sigma) \subset \text{st}(\tau) &\Leftrightarrow I_{\text{st}(\sigma)} \supset I_{\text{st}(\tau)} \\ &\Leftrightarrow P_{\neg x^a} \supset P_{\neg x^b} \\ &\Leftrightarrow P_{x^a} \subset P_{x^b}. \end{aligned}$$

Putting this together with 3.1, we have

$$\begin{aligned}
x^a \partial^{(b)} \in D(R) &\Leftrightarrow \forall \mathfrak{p} \in P : x^a \in \mathfrak{p} \vee x^b \notin \mathfrak{p} \\
&\Leftrightarrow \forall \mathfrak{p} \in P : \mathfrak{p} \in P_{x^a} \vee \mathfrak{p} \in P_{\neg x^b} \\
&\Leftrightarrow P = P_{x^a} \cup P_{\neg x^b} \\
&\Leftrightarrow P_{x^a} \subset P_{x^b} \vee P_{\neg x^b} \subset P_{\neg x^a} \text{ (and these are equivalent)} \\
&\Leftrightarrow st(\sigma) \subset st(\tau).
\end{aligned}$$

□

Example 3.5. Let $R = k[x_1, x_2, x_3, x_4, x_5]/(x_1x_3, x_1x_4, x_2x_4)$, the associated K is three 2-simplices $\{x_1, x_2, x_5\}, \{x_2, x_3, x_5\}, \{x_3, x_4, x_5\}$ glued along the edges $\{x_2, x_5\}$ and $\{x_3, x_5\}$; x_5 is a common vertex to all faces. Note that this makes K a simplicial join of $\{x_5\}$ with the complex from Example 3.3. Looking at the closed stars of the faces, we see that

$$st(x_1) \subset st(x_2) \subset st(x_5) \supset st(x_3) \supset st(x_4).$$

As $st(x_1) = st(\{x_1, x_2\})$, $st(x_4) = st(\{x_4, x_3\})$ and for any face σ , $st(\sigma) = st(\sigma \cup x_5)$ this accounts for all the stars. From this we should by 3.4 have the “toric” generators $x_i^a \partial_i^{(b)}$, and also $x_1^a \partial_2^{(b)}, x_1^a \partial_5^{(b)}, x_1^a \partial_2^{(b)} \partial_5^{(c)}, x_2^a \partial_5^{(b)}$ and the same with x_1 and x_2 replaced by x_4 and x_3 respectively (by symmetry). In fact, $st(x_5) = st(\emptyset) = K$, so we should also have $\partial_5^{(a)} = 1 \cdot \partial_5^{(a)}$ and the description is somewhat redundant.

From 3.4 we deduce the following very useful criterion.

Corollary 3.6. $\langle x_\tau \rangle \subset \langle x_\sigma \rangle$ if and only if $st(\tau) \subset st(\sigma)$.

Proof. If $st(\tau) \subset st(\sigma)$, it follows from 3.4 that $x_\tau \partial_\sigma = x_\tau \prod_{i: x_i \in \sigma} \partial_i$ is in $D(R)$. Now observe that $[\cdots [x_\tau \partial_\sigma, x_{i_1}], \cdots, x_{i_r}] = x_\tau$ (where $x_\sigma = \prod_{1 \leq j \leq r} x_{i_j}$), so we have $x_\tau \in \bigcap_{i: x_i \in \sigma} \langle x_i \rangle = \langle x_\sigma \rangle$.

To show the reverse implication, note that by definition of $I_{st(\sigma)}$, we have $I_{st(\sigma)} \cap \langle x_\sigma \rangle = \langle 0 \rangle$. If now $st(\tau) \not\subset st(\sigma)$, it follows that $\tau \in U_\sigma$, so $x_\tau \in I_{st(\sigma)}$, which finally implies $x_\tau \notin \langle x_\sigma \rangle$. □

The following very useful result is surprising.

Theorem 3.7. Any proper two-sided ideal in $D(R)$ is generated by reduced monomials in the “ordinary” variables x_1, \dots, x_n .

Proof. The proof is in three parts:

- (1) The ideal $\langle \sum_{(a,b) \in S} c_{ab} x^a \partial^{(b)} \rangle$ (for some index set $S \subset \mathbb{N}^{2n}$) is equal to the ideal $\langle x^a \partial^{(b)} \mid (a,b) \in S \rangle$;
- (2) the ideal $\langle x^a \rangle$ is equal to the ideal $\langle \prod_{a_i \neq 0} x_i \rangle$;
- (3) the ideal $\langle x^a \partial^{(b)} \rangle$ is equal to the ideal $\langle \prod_{a_i \neq 0} x_i \rangle$.

We will make heavy use of the fact that for any two-sided ideal I and any element $\phi \in D(R)$, the set of commutators $[\phi, I]$ is contained in I .

For the first part, recall that we have two natural concepts of grading on the Weyl algebra, that descend to $D(R)$. First, the natural \mathbb{Z}^n -grading on the Weyl algebra given by the *degree*

$$deg(x^a \partial^{(b)}) = (a_1 - b_1, \dots, a_n - b_n),$$

which induces a grading on $D(R)$; second we have the \mathbb{N}^n -grading given by the order

$$\text{ord}(x^a \partial^{(b)}) = (b_1, \dots, b_n).$$

Note that

$$\begin{aligned} [x_i \partial_i, x^a \partial^{(b)}] &= x_i \partial_i x^a \partial^{(b)} - x^a \partial^{(b)} x_i \partial_i \\ &= x_i (x^a \partial_i + a_i x^{a-1_i}) \partial^{(b)} - x^a (x_i \partial^{(b)} + \partial^{(b-1_i)}) \partial_i \\ &= x_i x^a \partial_i \partial^{(b)} + a_i x_i x^{a-1_i} \partial^{(b)} - x^a x_i \partial^{(b)} \partial_i - x^a \partial^{(b-1_i)} \partial_i \\ &= a_i x^a \partial^{(b)} - \binom{b_i - 1 + 1}{1} x^a \partial^{(b)} \\ &= (a_i - b_i) x^a \partial^{(b)} \end{aligned}$$

(in the remainder we omit the proof of such identities to avoid tedium), and in the case of characteristic p , if $a_i - b_i = cp^r$, we have $[x_i^{p^r} \partial_i^{(p^r)}, x^a \partial^{(b)}] = cx^a \partial^{(b)}$. In other words, the operators $[x_i \partial_i, -]$ (and $[x_i^{p^r} \partial_i^{(p^r)}, -]$) give different weight to each degree-graded component. Note also that

$$[x^a \partial^{(b)}, x_i] \cdot \partial_i = x^a \partial^{(b-1_i)} \partial_i = b_i x^a \partial^{(b)},$$

and if $b_i = cp^r$, we have $[x^a \partial^{(b)}, x_i^{p^r}] \partial_i^{(p^r)} = cx^a \partial^{(b)}$. In other words the operators $[-, x_i] \partial_i$ (and $[-, x_i^{p^r}] \partial^{p^r}$) give different weight to each order-graded component. Putting these together, we can isolate any term $x^a \partial^{(b)}$ by applying a suitable polynomial in the operators $[x_i \partial_i, -]$, $[x_i^{p^r} \partial_i^{(p^r)}, -]$, $[-, x_i] \partial_i$ and $[-, x_i^{p^r}] \partial^{p^r}$.

For the second part, we may reduce to a single variable. We separate the cases by characteristic. If $\text{char}(k) = p$, we have $[x_i \partial_i^{(p^r)}, x_i^{p^r}] = x_i$, so x_i is in the ideal generated by $x_i^{p^r}$; choosing a power of p larger than a_i we have $x_i^{p^r} = x_i^{a_i} \cdot x_i^{p^r - a_i}$ and so $x_i \in \langle x_i^{a_i} \rangle$. If $\text{char}(k) = 0$, on the other hand, we have

$$[x_i \partial_i^{(2)}, x_i^{a_i}] = a_i x_i^{a_i} \partial_i + \binom{a_i}{2} x_i^{a_i-1}$$

and

$$[x_i^2 \partial_i^{(3)}, x_i^{a_i}] = a_i x_i^{a_i+1} \partial_i^{(2)} + \binom{a_i}{2} x_i^{a_i} \partial_i + \binom{a_i}{3} x_i^{a_i-1}.$$

If $a_i = 0, 1$ there is nothing to prove, and if $a_i > 1$, we can invert $\frac{a_i-1}{2} \binom{a_i}{2} - \binom{a_i}{3} = \frac{1}{12} a_i (a_i^2 - 1)$ and get

$$x_i^{a_i-1} = \frac{12}{a_i(a_i^2-1)} \left(\frac{a_i-1}{2} [x_i \partial_i^{(2)}, x_i^{a_i}] - [x_i^2 \partial_i^{(3)}, x_i^{a_i}] + a_i x_i^{a_i} \cdot x_i \partial_i^2 \right).$$

This gives $\langle x_i^{a_i-1} \rangle \subset \langle x_i^{a_i} \rangle$ and by iterating this procedure, $\langle x_i \rangle = \langle x_i^{a_i} \rangle$.

For the third part, observe that $[x^n \partial^{(m)}, x_j] = x^n \partial^{(m-1_j)}$ (for j such that $m_j \neq 0$) is a valid identity for all $n, m > 0$. Iterating this beginning with $n = a, m = b$ gives $\langle x^a \rangle \subset \langle x^a \partial^{(b)} \rangle$. By applying part 2 this becomes $\langle \prod_{a_i \neq 0} x_i \rangle \subset \langle x^a \partial^{(b)} \rangle$.

To show the reverse implication $\langle x^a \partial^{(b)} \rangle \subset \langle \prod_{a_i \neq 0} x_i \rangle$ we show $\langle x^a \partial^{(b)} \rangle \subset \langle x_i \rangle$ for the two cases $a_i, b_i \neq 0$ and $a_i \neq 0, b_i = 0$. For the first case, $x_i^{a_i} \partial_i^{(b_i)}$ is a factor of $x^a \partial^{(b)}$; and applying the above argument we have that $x_i^{a_i} \partial_i^{(b_i)} \in \langle x_i \rangle$; it follows that $x^a \partial^{(b)} \in \langle \prod_{i: a_i, b_i \neq 0} x_i \rangle$.

For the second case, $a_i \neq 0, b_i = 0$, we may assume $a_i = 1$, for if $a_i > 1$, then clearly $x^a \partial^{(b)} = x_i x^{a-1} \partial^{(b)} \in \langle x_i \rangle$. By the previous case, $x_i \partial_i^{(2)}$ is in $\langle x_i \rangle$, and so is $x^{a+1} \partial^{(b)} = x_i x^a \partial^{(b)}$; then of course their commutator

$$[x^{a+1} \partial^{(b)}, x_i \partial_i^{(2)}] = -(a_i + 1)x^{a+1} \partial_i \partial^{(b)} - a_i x^a \partial^{(b)}$$

is also in $\langle x_i \rangle$. Rewriting this (with $a_i = 1$ as we have assumed) we get

$$x^a \partial^{(b)} = [x^{a+1} \partial^{(b)}, x_i \partial_i^{(2)}] - 2x^{a+1} \partial_i \partial^{(b)}$$

and so $x^a \partial^{(b)} \in \langle x_i \rangle$; it follows that $x^a \partial^{(b)} \in \langle \prod_{i: a_i \neq 0, b_i = 0} x_i \rangle$. Taking both cases together we have shown that $x^a \partial^{(b)} \in \langle \prod_{i: a_i \neq 0, b_i = 0 \vee b_i \neq 0} x_i \rangle = \langle \prod_{i: a_i \neq 0} x_i \rangle$. \square

We have shown that all ideals in $D(R)$ are generated by reduced monomials $\prod x_i$ in the variables of R ; the next question is of course which ones? Recall that we will not distinguish between the vertices of the simplicial complex K and the variables of the associated face ring R , but refer to either by the same name, e.g. x_i . We also remind of the notation $x_\sigma = \prod_{x_i \in \sigma} x_i$.

Theorem 3.8. *Any proper ideal in $D(R)$ is generated by monomials x_σ with $\sigma \in K$ such that $st(\sigma) \neq K$.*

Proof. From 3.7 it follows that any ideal in $D(R)$ is generated by reduced monomials in the variables x_i , and clearly the monomials corresponding to non-faces cannot occur as they are in I_K , so what remains are the monomials x_σ for $\sigma \in K$. Only those x_σ such that $st(\sigma) \neq K$ generate proper ideals, as otherwise we have $st(\sigma) = K$ and by 3.4 the elements $1 \cdot \partial_i$ where $x_i \in \sigma$ are in $D(R)$, as both 1 and ∂_i are monomials with support contained in $st(\sigma) = K$; if we write $\sigma = \{x_{i_1}, \dots, x_{i_t}\}$, we have $[\partial_{i_1}, [\partial_{i_2}, [\dots, [\partial_{i_t}, x_\sigma] \dots]]] = 1$ and so $\langle x_\sigma \rangle = \langle 1 \rangle = R$. \square

This now gives us all the ideals in $D(R)$, as by sums of principal ideals $\langle x_\sigma \rangle$ we can make everything. We may however also take a different approach: Any two-sided ideal in $D(R)$ is the kernel of some ring homomorphism; the combinatorial structure of the associated simplicial complex K gives rise to several such maps. An obvious choice for candidate homomorphisms is the localization at an element x_σ ; we will see that the kernels of such maps is another generating set for the lattice of two-sided ideals in $D(R)$. We introduce the notation \overline{J} for the extension to $D(R)$ of an ideal $J \subset R$.

Theorem 3.9. *The kernel of the localization map $D(R) \rightarrow D(R)[\frac{1}{x_\sigma}]$ is the extension $\overline{I_{st(\sigma)}}$ of the ideal $I_{st(\sigma, K)} \subset R$ to $D(R)$.*

Proof. By 3.8 it is enough to examine what happens in the localization to monomials x_α for $\alpha \in K$. Assume first that $x_\sigma = x_i$ (in other words, σ is a vertex). Inverting x_i has the effect that for any non-face $\beta = \cup x_j$ containing x_i , the monomial $\frac{x_\beta}{x_i} = \prod_{x_j \in \beta, j \neq i} x_j$ is zero in the localization. It is clear that no other monomials are killed, so what remains after localization are those monomials supported on a face τ such that $\tau \cup x_i$ is not a non-face, or clearing negations, that $\tau \cup x_i$ is a face in K ; in other words the remaining monomials are those supported on a face of $st(x_i)$.

For the general case, note that inverting $x_\sigma = \prod_i x_i$ is the same as inverting each x_i successively, and observing that we have from 2.3(iii) that $st(\sigma, K) = st(x_1, st(\sigma \setminus x_1, K))$, we are done by recursion. \square

Theorem 3.10. *The lattice of two-sided ideals in $D(R)$ is generated by the ideals $\overline{I_{st(\sigma)}} \subset D(R)$.*

Proof. After applying 3.8 the question is whether we can generate any proper ideal $\langle x_\tau \rangle$ by sums and intersections of the ideals $\overline{I_{st(\sigma)}}$. Considering that $\overline{I_{st(\sigma)}} = \langle x_\alpha | \alpha \in U_\sigma \rangle$, we can look at the intersection of all such ideals that contain x_τ :

$$\begin{aligned}
 (3.1) \quad \bigcap_{\sigma: \tau \in U_\sigma} \overline{I_{st(\sigma)}} &= \langle x_\alpha | \alpha \in \bigcap_{\sigma: \tau \in U_\sigma} U_\sigma \rangle \\
 (3.2) \quad &= \langle x_\alpha | \forall \sigma \in K : \tau \in U_\sigma \Rightarrow \alpha \in U_\sigma \rangle \\
 (3.3) \quad &= \langle x_\alpha | \forall \sigma \in K : \alpha \notin U_\sigma \Rightarrow \tau \notin U_\sigma \rangle \\
 (3.4) \quad &= \langle x_\alpha | \forall \sigma \in K : \alpha \cup \sigma \in K \Rightarrow \tau \cup \sigma \in K \rangle \\
 (3.5) \quad &= \langle x_\alpha | \forall \sigma \in K : \sigma \in st(\alpha) \Rightarrow \sigma \in st(\tau) \rangle \\
 (3.6) \quad &= \langle x_\alpha | st(\alpha) \subset st(\tau) \rangle \\
 (3.7) \quad &= \langle x_\tau \rangle
 \end{aligned}$$

where the last step is applying Corollary 3.6. \square

Example 3.11. Consider again the ring from 3.3, $R = k[x_1, x_2, x_3, x_4]/I$ where $I = (x_1x_3, x_1x_4, x_2x_4)$; the associated complex K is a chain of three 1-simplices. Inverting x_1 gives us that x_3 and x_4 go to zero in the localization as $x_3 = \frac{1}{x_1}x_1x_3 \in I$, etc; it follows that the generators $x_4^a \partial_3^{(b)}$ are also killed; the kernel of the localization $D(R) \rightarrow D(R)[\frac{1}{x_1}]$ is then (using 3.7 and 3.4) the ideal (x_3, x_4) , which is the face ideal of $st(x_1, K)$. Localizing at x_2 gives $x_3 = \frac{1}{x_2}x_2x_3 = 0$, and the kernel of the localization is indeed equal to the ideal (x_4) , the face ideal of $st(x_2, K)$. Proceeding in the same manner for the remaining faces $x_3, x_2, \{x_1, x_2\}, \{x_2, x_3\}$, and $\{x_3, x_4\}$, we get as possible kernels the ideals $(x_1), (x_4), (x_1, x_2), (x_3, x_4)$ and (x_2, x_3) . By 3.4 we have $(x_1, x_2) = (x_2)$ and $(x_3, x_4) = (x_3)$; in other words our possible kernels of localization are the ideals $(x_1), (x_2), (x_3)$ and (x_4) ; in light of 3.7 these obviously generate all the ideals by sums and intersections.

Let us round off this section with some applications. In [Tra99], Traves examines the $D(R)$ -module structure of R when k is a field, and determines what the (left) $D(R)$ -submodules of R are. These are the ideals $I \subset R$ such that $D(R) \bullet I = I$, so we follow Traves' terminology and call such a submodule a $D(R)$ -stable ideal. The reason for restricting k to be a field is that elements of $D_k(R)$ are k -linear endomorphisms of R , so any ideal of k extends to a $D_k(R)$ -submodule of R .

Theorem 3.12 (Traves). *When k is a field, the $D_k(R)$ -submodules of the reduced monomial ring R are exactly the ideals given by intersections of sums of minimal primes of R .*

Based on our results about the ideal structure of $D(R)$, we can give a new proof of this result. We denote the module action of $D(R)$ by \bullet (e.g. $D(R) \bullet I$) and the product in $D(R)$ by \cdot (e.g. $D(R) \cdot I$). We prove the result by means of a general fact which to our knowledge is previously unknown.

Proposition 3.13. *Let k be a field and R be a k -algebra. An ideal $J \subset R$ is $D(R)$ -stable if and only if $J = \overline{J} \cap R$, where \overline{J} denotes the extension of J to $D(R)$.*

Proof. Observe first that R is isomorphic as a $D(R)$ -module to $D(R)/D^{>0}(R)$, the quotient by the left ideal of positive order elements; we can see this by writing $D(R) = D^0(R) + D^{>0}(R) = R + D^{>0}(R)$, as $R = D^0(R)$. In other words, if $S \subset R$ is a subset, then under this isomorphism $D(R) \bullet S = D(R) \cdot S + D^{>0}(R)$. Further, if $J \in D(R)$ is some subset, then

$$\begin{aligned} J \cdot D(R) + D^{>0}(R) &= J \cdot (D(R)^0 + D^{>0}(R)) + D^{>0}(R) \\ &= J \cdot R + D^{>0}(R). \end{aligned}$$

Now, if $I \subset R$ is an ideal, the extension of I to $D(R)$ is $\bar{I} = D(R) \cdot I \cdot D(R)$, so we have

$$\begin{aligned} \bar{I} + D^{>0}(R) &= D(R) \cdot I \cdot D(R) + D^{>0}(R) \\ &= D(R) \cdot I + D^{>0}(R) \\ &= D(R) \bullet I. \end{aligned}$$

A D -stable ideal is an ideal $I \subset R$ such that $D(R) \bullet I = I$, so it follows that the D -stable ideals are exactly those such that $\bar{I} + D^{>0}(R) = I$.

It remains to show that for an ideal $J \subset D(R)$, $J + D^{>0}(R) = J \cap R$. Let $f \in J$ be some element, and write it as the sum $f = f_0 + f_1 + \cdots + f_{\text{ord}(f)}$ where f_i are the terms of order i ; it then follows from 3.7 that also each $f_i \in J$. Reducing modulo $D^{>0}(R)$ we get $J + D^{>0}(R) = \{f_0 | f \in J\}$, and restricting to the homogenous elements of order zero we have $J \cap R = J \cap D^0(R) = \{f \in J | f = f_0\}$; these sets clearly are equal. \square

Theorem 3.14. *The $D(R)$ -stable ideals of R are those generated by sums and intersections of the ideals $I_{st(\sigma)}$ for $\sigma \in K$.*

Proof. As we have shown (3.8, 3.10) that any ideal of $D(R)$ is an extension of an ideal of R , we only have to restrict these to R to recover the $D(R)$ -stable ideals. Theorem 3.10 tells us that the lattice of ideals in $D(R)$ is generated by sums and intersections of ideals $\overline{I_{st(\sigma)}}$, and it is easy to see that $\overline{I_{st(\sigma)}} \cap R = I_{st(\sigma)}$: Indeed, the only possible problem is that in $D(R)$, $\langle x_\alpha \rangle \subset \langle x_\beta \rangle$ if and only if $st(\alpha) \subset st(\beta)$, and this may cause additional monomials not in I to appear in $\bar{I} \cap R$. For $I_{st(\sigma)}$ however, this does not happen. Consider that $I_{st(\sigma)} = \langle x_\tau | \tau \in U_\sigma \rangle$ and $\overline{I_{st(\sigma)}} \cap R = \langle x_\tau | \tau \in U_\sigma \rangle = \langle x_\alpha | \exists \tau \in U_\sigma : st(\alpha) \subset st(\tau) \rangle$. In other words, we need to check if there are faces $\tau \in U_\sigma$ and $\alpha \in st(\sigma)$ such that $st(\alpha) \subset st(\tau)$, as then x_α would be in $\overline{I_{st(\sigma)}} \cap R$, but not in $I_{st(\sigma)}$. This is impossible, however: by 2.3(v), $\alpha \in st(\sigma)$ if and only if $\sigma \in st(\alpha)$, and if $st(\alpha) \subset st(\tau)$, we have $\sigma \in st(\tau)$, which again by 2.3(v) gives $\tau \in st(\sigma)$, which contradicts the assumption $\tau \in U_\sigma$. \square

To recover 3.12, recall that by 2.4, the minimal primes are exactly the face ideals of the maximal faces of K , and any $I_{st(\sigma)}$ is the intersection of the face ideals of the maximal faces of $st(\sigma)$.

Remark 3.15. Recall that the partially ordered set of two-sided ideals of $D(R)$ (or bijectively, the $D(R)$ -stable ideals of R) is in order-reversing bijection with the partially ordered set of closed stars of K . This partially ordered set can be completed to a simplicial complex (\tilde{K} , say), homotopic to the nerve of the cover of K by open stars. The results about two-sided ideals of $D(R)$ and $D(R)$ -stable ideals of R imply that subcomplexes L of K such that I_L is $D(R)$ -stable or \bar{I}_L is a

two-sided ideal of $D(R)$ are exactly those that are unions of intersections of closed stars; in other words the complex \tilde{K} classifies such subcomplexes. This interesting connection is perhaps worthy of further study.

4. CHARACTERISTIC p

The constructions in the previous section are independent of the characteristic of k , and so solve the problem of finding the two-sided ideal structure of $D(R)$. In characteristic p however, there is a qualitatively different construction of $D(R)$, which perhaps offers more interesting possibilities for generalization. From here on, we assume k is a field of characteristic p .

The major tool when working in characteristic p is the Frobenius automorphism of k , given by $x \mapsto x^p$. This induces an endomorphism $F : R \rightarrow R$ given by $F(f) = f^p$, and the image $F(R)$ is the subring $R^p \subset R$ of p 'th powers; as R is reduced F is also an isomorphism onto its image. Any R -module M gets a new R -module structure through the pullback by the Frobenius map, namely F_*M is equal to M as an abelian group, but has R -module structure given by $f \cdot m = f^p m$. This is equivalent to considering M as an R^p -module, as the maps $F : R \rightarrow R$ and $R^p \hookrightarrow R$ both are injections with image R^p . We will have need for considering also iterates of F , so if we let $q = p^r$ we write $F^r : R \rightarrow R$ or $R^q = R^{p^r} \subset R$. For our purposes in examining $D(R)$, it will be most convenient to use the description in terms of the subrings R^q , as we will see.

Considering the behaviour R itself as an R^p -module gives rise to several classifying properties of the ring R . We will simply recall the definitions of the particular properties that are relevant for us, other such properties and further details may be found in [SVdB97]. If R is finitely generated as an R^p -module, we say that R is *F-finite*; if R is *F-finite* and the map $R^p \hookrightarrow R$ splits as a map of R^p -modules we say R is *F-split*; if $F_*^r R \simeq M_1^r \oplus \cdots \oplus M_{n(r)}^r$ as an R -module and the set of isomorphism classes $\{[M_i^r] | r \in \mathbb{N}, 1 \leq i \leq n(r)\}$ of modules appearing in such a decomposition for some r is finite, we say that R has *finite F-representation type*, or *FFRT*.

For our purposes, the key property of face rings R_K in this respect is that they are *F-split* and have *FFRT*. Even better, we can give a concrete decomposition of R as an R^q -module:

Lemma 4.1. *As an R^q -module, R is isomorphic to $\bigoplus_{st(\sigma) \subset K} (R_{st(\sigma)}^q)^{m_{st(\sigma)}(q)}$, where $m_{st(\sigma)}(q) = \sum_{\alpha: st(\alpha)=st(\sigma)} (q-1)^{(\dim(\alpha)+1)}$.*

Note that the direct sum runs over those subcomplexes of K that is the star of some simplex.

Proof. As we have $R \simeq R^p \oplus R^p x_1 \oplus \cdots \oplus R^p x_1^{p-1} \cdots x_n^{p-1}$ (where only the appropriate monomials appear), this expresses R as an R^p -module. We can rewrite this using $R^p \cdot x^\alpha \simeq R^p / \text{Ann}_{R^p}(x^\alpha)$, and observing that as the monomials x^α that appear in the decomposition are those supported on a face $\text{supp}(\alpha) =: \sigma$, and that the annihilator of x^α is the face ideal of the complex $st(\sigma, K)$, we get the decomposition $R = \bigoplus_{\sigma \in K} (R_{st(\sigma)}^p)^{m_{st(\sigma)}(p)}$, where $R_{st(\sigma)}^p$ is the (p 'th power) face ring of $st(\sigma)$ and by simply counting monomials we have $m_{st(\sigma)}(p) = \sum_{\alpha: st(\alpha)=st(\sigma)} (p-1)^{(\dim(\alpha)+1)}$ (using the convention that $\dim(\emptyset) = -1$). Iterating the same construction, we get $R = \bigoplus_{\sigma \in K} (R_{st(\sigma)}^q)^{m_{st(\sigma)}(q)}$, where $m_{st(\sigma)}(q) = (q-1)^{(\dim(\sigma)+1)}$. \square

Let us make use of this to compute some invariants of R that only make sense in characteristic p , namely the *Hilbert-Kunz function* and the *Hilbert-Kunz multiplicity*. This invariant was introduced by Kunz [Kun69] for local rings, and extended to graded rings by Conca [Con96]; see also [Hun13] and [Mon83].

Definition 4.2. Let R be a local ring with maximal ideal \mathfrak{m} , or a graded ring with homogenous maximal ideal \mathfrak{m} , over a field k of characteristic p , and let $q = p^r$. The *Hilbert-Kunz function* of a ring R is the function

$$HK_R(q) = l(R/\mathfrak{m}^{[q]})$$

where $I^{[q]}$ is the ideal generated by q 'th powers of elements in the ideal I . The *Hilbert-Kunz multiplicity* is the number

$$e_{HK}(R) = \lim_{q \rightarrow \infty} \frac{HK_R(q)}{q^{\dim R}},$$

in other words the leading coefficient of $HK_R(q)$.

The Hilbert-Kunz function gives a measure of singularity of R , roughly speaking higher multiplicities correspond to worse singularities. It is a theorem of Kunz that $HK_R(q) = q^{\dim R}$ if and only if R is regular (see [Kun69]), so if R is regular, $e_{HK}(R) = 1$. The converse holds for *unmixed* rings, but not in general, and in particular not for face rings. The following is equivalent to Remark 2.2 in [Con96], though we prove it in a different way.

Proposition 4.3. Let R_K be a face ring, then $HK_R(q) = \sum_{i=-1}^{\dim(R)-1} f_i(q-1)^{i+1}$, where f_i is the number of i -simplices in K , so $(f_{-1}, \dots, f_{\dim(R)-1})$ is the f -vector of K (we recall the usual convention $\dim(\emptyset) = -1$, so $f_{-1} = 1$). In particular, $e_{HK}(R_K) = f_{\dim K}$, the number of top-dimensional faces of K .

Proof. The number of indecomposable summands of R as an R^q -module is $\sum_{\sigma \in K} (q-1)^{\dim(\sigma)+1}$ by 4.1. By simply rearranging the sum, this is equal to $\sum_{i=-1}^{\dim(R)-1} f_i(q-1)^{i+1}$. The claim now follows from the fact that none of the generators of these summands are in $\mathfrak{m}^{[q]} = \langle x_1^q, \dots, x_n^q \rangle$, so the number of summands in the splitting of R is the same as the length of $R/\mathfrak{m}^{[q]}$. \square

The promised different construction of $D(R)$ is due to Yekutieli [Yek92]. We omit the proof here, but mention that in addition to [Yek92], the reader can find an excellent exposition in [SVdB97].

Proposition 4.4. $D_k(R) \simeq \bigcup_q \text{End}_{R^q}(R)$, where $q = p^r, r \in \mathbb{N}$ and R^q is the subring of q -th powers.

Let us now give the summands appearing in 4.1 a more convenient notation, and define $M_{st(\sigma)}^q := (R_{st(\sigma)}^q)^{m_{st(\sigma)}(q)}$. It follows from 4.1 that

$$\text{End}_{R^q}(R) \simeq \bigoplus_{st(\sigma), st(\tau) \subset K} \text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q).$$

As each $M_{st(\sigma)}^q$ is generated as an R^q -module by monomials of degree in each variable up to $q-1$, we can see that as an R^{pq} -module it is contained in $\bigoplus_{st(\alpha) \subset st(\sigma)} M_{st(\alpha)}^{pq}$, because the elements of $M_{st(\sigma)}^q$ contain monomials of degree larger than $q-1$, which have support on smaller stars (recall that as $q = p^r$, $pq = p^{r+1}$). In particular this implies the following:

Lemma 4.5. $\text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q) \subset \bigoplus_{st(\alpha) \subset st(\sigma), st(\beta) \subset st(\tau)} \text{Hom}_{R^{pq}}(M_{st(\alpha)}^{pq}, M_{st(\beta)}^{pq})$.

This lets us think of elements $\phi \in \text{End}_{R^q}(R)$ as block matrices with each block having entries in some $R^q/I_{st(\sigma)}$; it is vital to remember that this means that the entries have degree equal to a multiple of q .

Definition 4.6. Let $J_q(st(\alpha), st(\beta))$ denote the ideal in $D(R)$ generated by the elements of $\text{Hom}_{R^q}(M_{st(\alpha)}^q, M_{st(\beta)}^q)$, and let $J(st(\alpha), st(\beta)) := \sum_q J_q(st(\alpha), st(\beta))$. For convenience we denote $J(st(\sigma), st(\sigma))$ by simply $J(st(\sigma))$.

The following result is essentially the same as 3.7 in a different guise.

Proposition 4.7. Assume $st(\sigma) \supset st(\tau)$, and let $\phi \in \text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q)$ be a nonzero element. Then $\langle \phi \rangle$, the ideal in $D(R)$ generated by ϕ , is equal to the ideal $J(st(\tau))$. Furthermore, we have that $J(st(\tau)) \subset J(st(\sigma))$.

Proof. Clearly, $J(st(\tau))$ is generated by the identity maps $id_{st(\tau)}^q : M_{st(\tau)}^q \rightarrow M_{st(\tau)}^q$ (for each q), so it suffices to show that these are in $\langle \phi \rangle$.

Recall that any element of $\text{End}_{R^q}(R)$ has entries with degree a multiple of q . We claim that for $s > q$ a sufficiently large power of p , ϕ considered as an element of $\text{End}_{R^s}(R)$ will have at least some constant entries in each block $\text{Hom}_{R^s}(M_{st(\sigma)}^s, M_{st(\tau)}^s)$. To see this, suppose ϕ (as an element of $\text{End}_{R^q}(R)$) has an entry x_i^q in a block $\text{Hom}_{R^q}(R^q \cdot x^a, R^q \cdot x^b)$ (with all $0 \leq a_j, b_j < q$), in other words $\phi(x^{a+cq}) = x^{a+(c+1_i)q+b}$. It follows from 4.5 that this block has image in $\text{End}_{R^{pq}}(R)$ contained in $\bigoplus_{0 \leq c, d < p} \text{Hom}_{R^{pq}}(R^{pq} \cdot x^{a+cq}, R^{pq} \cdot x^{b+dq})$, and as $\phi(x^{a+cq}) = x^{a+(c+1_i)q+b} = x^{a+cq+(b+1_i)q}$ this yields the entry 1 in the blocks $\text{Hom}_{R^{pq}}(R^{pq} \cdot x^{a+cq}, R^{pq} \cdot x^{b+1_iq})$. In similar fashion an entry with degree nq will yield constant entries somewhere when considered as an R^s -linear map for $s > q$ a sufficiently large power of p .

Now let s be such a sufficiently large power of p , and consider ϕ as an element of $\text{End}_{R^s}(R)$; by 4.5, $\text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q)$ is contained in $\bigoplus_{st(\alpha) \subset st(\sigma), st(\beta) \subset st(\tau)} \text{Hom}_{R^s}(M_{st(\alpha)}^s, M_{st(\beta)}^s)$. We can see that ϕ , considered as a matrix (ϕ_{ij}) in $\text{End}_{R^s}(R)$, will have (among others) some constant entries in each block $\text{End}_{R^s}(M_{st(\beta)}^s)$ such that $st(\beta) \subset st(\tau)$. Each of these entries can be “picked out” in the following manner: Let $\mathbf{1}_{ii}$ be the matrix in $\text{End}_{R^s}(R)$ with the appropriate identity map in position (i, i) and zeroes otherwise. It is clear that $\mathbf{1}_{ii} \cdot \phi \cdot \mathbf{1}_{jj}$ is the matrix with entry ϕ_{ij} in position (i, j) and zeroes otherwise; we may assume $\phi_{ij} = 1$ as it is constant. Applying permutations of $\text{End}_{R^s}(M_{st(\beta)}^s)$ (on both sides), we can now place this entry 1 wherever we want within the matrix block corresponding to $\text{End}_{R^s}(M_{st(\beta)}^s)$; taking sums of these we can produce any matrix with constant entries. In particular, we can make $id_{st(\beta)}^s$.

Thus, we have that each $id_{st(\beta)}^s$ such that $st(\beta) \subset st(\tau)$ is in $\langle \phi \rangle$, and in the same way any such $id_{st(\beta)}^t$ for $t > s$ any larger power of p . To recreate $id_{st(\tau)}^t$ for smaller powers $t < s$ we observe that those maps, considered as elements of $\text{End}_{R^s}(R)$, are in $\bigoplus_{st(\beta) \subset st(\tau)} \text{End}_{R^s}(M_{st(\beta)}^s)$ and as such are contained in the ideal generated by the identity maps $id_{st(\beta)}^s$, in other words contained in $\langle \phi \rangle$. We have shown $J(st(\tau)) \subset \langle \phi \rangle$; the opposite inclusion follows from the observation that $\phi = id_{st(\tau)}^q \circ \phi$, and so $\phi \in J(st(\tau))$.

The final claim is similar: $\phi = \phi \circ id_{st(\sigma)}^q$, and so $\phi \in J(st(\sigma))$. \square

Proposition 4.8. The ideal $J(st(\sigma), st(\tau))$ is equal to $J(st(\sigma \cup \tau))$, if $\sigma \cup \tau$ is a face of K , and the zero ideal otherwise.

Proof. The module $\text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q)$ has support $st(\sigma)^\circ \cap st(\tau)^\circ$. From 2.3(vi) it follows that this is $st(\sigma \cup \tau)^\circ$, if $\sigma \cup \tau \in K$.

If $\sigma \cup \tau$ is a non-face, $st(\sigma) \cap st(\tau)$ does not contain any maximal simplices, and so the cone on $st(\sigma) \cap st(\tau)$ is not a union of irreducible components of $\text{Spec}(R)$, and so is not the closure of the support of any element in $\text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q)$, so this must be the zero module. It follows that $J(st(\sigma), st(\tau))$ is the zero ideal.

For the case when $\sigma \cup \tau$ is a face of K , recall that by Lemma 4.5,

$$\text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q) \subset \bigoplus_{st(\alpha) \subset st(\sigma), st(\beta) \subset st(\tau)} \text{Hom}_{R^{pq}}(M_{st(\alpha)}^{pq}, M_{st(\beta)}^{pq}).$$

In particular, there will be entries in the block $\text{Hom}_{R^{pq}}(M_{st(\sigma \cup \tau)}^{pq}, M_{st(\sigma \cup \tau)}^{pq})$, so by 4.7 we have that $J(st(\sigma \cup \tau)) \subset J(st(\sigma), st(\tau))$.

For the converse, note that as an R^q -module,

$$\text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q) \simeq ((I_{st(\tau)}^q : I_{st(\sigma)}^q) / I_{st(\tau)}^q)^{m_{st(\sigma)}(q) \times m_{st(\tau)}(q)}$$

(where I^q is the restriction of $I \subset R$ to R^q). Any element of $\text{Hom}_{R^q}(M_{st(\alpha)}^q, M_{st(\beta)}^q)$ has, as a matrix, entries with degree (in each variable) a multiple of q , with constant (nonzero) entries only when $st(\beta) \subset st(\alpha)$, as then $(I_{st(\beta)}^q : I_{st(\alpha)}^q)$ is the unit ideal in R^q (otherwise it is generated by elements of degree $\geq q$). It follows that elements of the image of $\text{Hom}_{R^q}(M_{st(\sigma)}^q, M_{st(\tau)}^q)$ in $\text{End}_{R^s}(R)$ for $s > q$ (considered as matrices) have entries with degree some multiple of s , with constant (nonzero) entries only in those blocks $\text{Hom}_{R^s}(M_{st(\alpha)}^s, M_{st(\beta)}^s)$ with $st(\beta) \subset st(\alpha)$. In the direct limit, these elements become infinite matrices with entries in k , in other words there can only be nonzero entries in those blocks corresponding to $st(\beta) \subset st(\alpha)$ (any nonzero entry in a different block must have infinite degree, which is impossible). This implies that $J(st(\sigma), st(\tau))$ is contained in $\sum_{st(\sigma) \supset st(\alpha) \supset st(\beta) \subset st(\tau)} J(st(\alpha), st(\beta))$, which by 4.7 is equal to $\sum_{st(\sigma) \supset st(\beta) \subset st(\tau)} J(st(\beta)) = J(st(\sigma \cup \tau))$ and we are done. \square

Theorem 4.9. *The ideals $J(st(\sigma))$ generate the lattice of ideals in $D(R)$ by sums and intersections.*

Proof. Let I be an ideal in $D(R)$; it is of course true in general that $I = \sum_{\phi \in I} \langle \phi \rangle$. By 4.7 and 4.8 this is equal to $\sum J(st(\sigma))$, where the sum goes over all $\sigma \in K$ such that I contains elements from some $\text{Hom}_{R^q}(M_{st(\alpha)}^q, M_{st(\sigma)}^q)$.

Finally, the intersection $J(st(\sigma)) \cap J(st(\tau))$ contains elements in those $\text{End}_{R^q}(M_{st(\alpha)}^q)$ with $st(\alpha) \subset st(\sigma) \cap st(\tau)$; the maximal such star is $st(\sigma \cup \tau)$ if $\sigma \cup \tau$ is a face of K , and if $\sigma \cup \tau$ is not a face, there are no such α ; in other words $J(st(\sigma)) \cap J(st(\tau)) = J(st(\sigma \cup \tau))$. \square

We have now given two essentially different descriptions of the ideals of $D(R)$, and we may wonder how to translate between the two languages. This is not too hard, as the obvious suggestion turns out to be true.

Theorem 4.10. *The ideal $J(st(\sigma))$ is equal to the ideal $\langle x_\sigma \rangle$.*

Proof. It follows from 4.7 and 4.8 that $J(st(\sigma)) = \bigoplus_{q > 0, st(\beta) \subset st(\sigma)} \text{Hom}_{R^q}(M_{st(\alpha)}^q, M_{st(\beta)}^q)$, in other words all the endomorphisms with support contained in $st(\sigma)$. We can think of x_σ as an endomorphism of R , given by $f \mapsto fx_\sigma$, and considering that whatever element f we choose, fx_σ has support contained in $st(\sigma)$. This means that the endomorphism x_σ is in $J(st(\sigma))$ and not in any larger ideal, and as $x_\sigma(1) = x_\sigma$ has

support equal to $st(\sigma)^\circ$, it is not in any smaller ideal $J(st(\tau))$ with $st(\tau) \subset st(\sigma)$. From 4.7 it follows that x_σ generates all of $J(st(\sigma))$ and the two ideals are equal. \square

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